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DISTRIBUTION OF EIGENVALUES FOR SOME SETS OF RANDOM MATRICES

V. A. MARČENKO AND L. A. PASTUR

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In this paper we study the distribution of eigenvalues for two sets of random Hermitian matrices and one set of random unitary matrices. The statement of the problem as well as its method of investigation go back originally to the work of Dyson [1] and I. M. Lifšic [2], [3] on the energy spectra of disordered systems, although in their probability character our sets are more similar to sets studied by Wigner [4].

Since the approaches to the sets we consider are the same, we present in detail only the most typical case. The corresponding results for the other two cases are presented without proof in the last section of the paper.

§1. Statement of the problem and survey of results

We shall consider as acting in N-dimensional unitary space H_N, a selfadjoint operator $B_N(n)$ of the form

$$B_N(n) = A_N + \sum_{i=1}^{n} \tau_i q^{(i)}(\cdot, q^{(i)}). \tag{1.1}$$

Here A_N is a nonrandom selfadjoint operator; n is a nonrandom number; the τ_i are independent identically distributed real random variables and the q^i are mutually independent random vectors in H_N, independent also of the τ_i.

The operators $q^{(i)}(\cdot, q^{(i)}) = L_i$ act on vectors $x \in H_N$ according to the formula $L_i(x) = q^{(i)}(x, q^{(i)})$, where $(x, q^{(i)})$ is the scalar product in H_N.

Thus the operators $B_N(n)$ we consider are sums of a nonrandom operator and a number of independent random one-dimensional operators. Each set of numbers $\tau_1, \cdots, \tau_n$ and vectors $q^{(1)}, \cdots, q^{(n)}$, which for brevity we denote by T_n, Q_n, gives a realization of the random operator $B_N(n)$.

We shall be interested in the function $\nu(\lambda;\ B_N(n))$ giving the ratio of the number of eigenvalues of $B_N(n)$ lying to the left of λ to the dimension of the space. From now on we call this the normalized spectral function of the operator. It is clear that for any realization T_n, Q_n the function $\nu(\lambda;\ B_N(n))$ is a nondecreasing left continuous and piecewise constant function of λ, and $0 \le \nu(\lambda;\ B_N) \le 1$.

For fixed λ the function $\nu(\lambda;\ B_N(n))$ is a random quantity determined in a complicated manner by the random numbers $\tau_1, \cdots, \tau_n$ and random vectors $q^{(1)}, \cdots, q^{(n)}$. The search for the probability distribution of this random quantity is one of the fundamental problems in the spectral analysis of random operators. Of particular interest is the case of very large N and n, since it often appears that for $N \to \infty$ the random quantity $\nu(\lambda;\ B_N(n))$ converges in probability to a nonrandom number.

We assume the following conditions are satisfied for $N \to \infty$.

I. The limit $\lim_{N\to\infty} n/N = c$, which for brevity we call the *concentration*, exists.

II. The sequence of normalized spectral functions $\nu(\lambda;\ A_N)$ of the operators A_N converges to some function $\nu_0(\lambda)$ at all points of continuity:

$$\lim_{N\to\infty} \nu(\lambda;\ A_N) = \nu_0(\lambda). \tag{1.2}$$

Assuming that these conditions are satisfied, it is necessary, first of all, to make clear how the stochastic properties of operators $B_N(n)$ of the form in (1.1) ensure the convergence in probability of the sequences $\nu(\lambda;\ B_N(n))$ to nonrandom numbers, i.e. to explain when a nondecreasing function $\nu(\lambda;\ c)$ exists such that at all of its points of continuity

$$\lim_{N\to\infty} P\{|\nu(\lambda;\ B_N(n)) - \nu(\lambda;\ c)| > \varepsilon\} = 0, \tag{1.3}$$

irrespective of $\epsilon > 0$.

The main problem for such a set of random operators consists, finally, in discovering the limit function $\nu(\lambda;\ c)$.

The case of physical interest is when all the τ_i are equal to the nonrandom variable τ and the vectors $q^{(i)}$ are selected from a given orthonormal system $e^{(1)}, \cdots, e^{(N)}$ with equal probabilities. In [2] and [3] I. M. Lifšic worked out a method for the approximate calculation of $\nu(\lambda;\ c)$ for small values of c in this case.

We point out one specific peculiarity of this case. Since the random vectors $q^{(i)}$ are chosen from a given orthonormal basis, the problem is not invariant under unitary transformations of the operator A_N, and therefore the answer depends not only on the normalized spectral functions $\nu(\lambda;\ A_N)$ of this operator, but also on all its eigenvectors. This remains the case as $N \to \infty$.

In this paper we consider another type of problem which as $N \to \infty$ becomes invariant with respect to unitary transformations of A_N. For the formulation of the conditions to be placed on the random vectors, it is convenient to select in the space H_N any orthonormal basis $e_1, \cdots, e_N$ and express the vectors in this coordinate system, writing $q = (q_1, \cdots, q_N)$, where $q_j = (q,\ e_j)$. In the remainder of this paper we assume the following conditions are satisfied.

III. The stochastic vectors $q = (q_1, \cdots, q_N)$ which enter into (1.1) have absolute moments to fourth order, inclusively, and the even moments $Mq_i\bar{q}_j$, $Mq_i\bar{q}_j q_l q_m$ can be put in the form[1)]

$$Mq_i\bar{q}_j = N^{-1}\delta_{ij} + a_{ij}(N), \tag{1.4}$$

$$Mq_i\bar{q}_j q_l \bar{q}_m = N^{-2}(\delta_{ij}\delta_{lm} + \delta_{im}\delta_{jl}) + \varphi_{il}(N)\overline{\varphi_{jm}(N)} + b_{ijlm}(N), \tag{1.5}$$

where $\delta_{\alpha\beta}$ is the Kronecker symbol and the quantities

1) We denote the expectation of the stochastic variable x by Mx.

$$\varepsilon_1(N)=\left[N\sum_{i,j}|a_{ij}(N)|^2\right]^{\frac{1}{2}};\ \varepsilon_2(N)=\sum_{i,j}|\varphi_{ij}(N)|^2;\ \varepsilon_3(N)=N\left[\sum_{i,j,l,m}|b_{ijlm}(N)|^2\right]^{\frac{1}{2}} \quad (1.6)$$

tend to infinity as $N \longrightarrow \infty$.

IV. The stochastic variables τ_i which enter into (1.1) are independent and have the same probability distribution $\sigma(x)$.

It is not difficult to prove that condition III on the stochastic vectors q is unitarily invariant, i.e. if they are satisfied for some one choice of orthonormal basis in H_N $(N = 1, 2, \cdots)$ then they are satisfied for any choice of basis.

The most typical example of stochastic vectors satisfying III is the set consisting of all unit vectors of H_N with each assigned the same probability (i.e. uniformly distributed on the unit sphere). In this case

$$Mq_i\bar{q}_j = N^{-1}\delta_{ij}, \quad Mq_i\bar{q}_jq_l\bar{q}_m = \frac{1}{N(N+1)}\{\delta_{ij}\delta_{lm}+\delta_{im}\delta_{jl}\},$$

which clearly ensures that III is satisfied.

We present two other examples of sets of random vectors for which III is satisfied.

a) The set of real unit vectors whose probability density $p(q_1, \cdots, q_N)$ is a symmetric and even function of all its arguments. For such vectors

$$Mq_iq_j = N^{-1}\delta_{ij}\,\text{and}\,Mq_iq_jq_lq_m = a_1(\delta_{ij}\delta_{lm}+\delta_{il}\delta_{jm}+\delta_{im}\delta_{jl})+(a_2-3a_1)\delta_{ij}\delta_{lm}\delta_{il},$$

where $a_1 = Mq_1^2q_2^2 = Mq_i^2q_k^2(i \neq k)$, $a_2 = Mq_1^4 = Mq_i^4$ $(i = 1, \cdots, N)$. By making use of the relation $1 = Na_2 + N(N-1)a_1$, following from the normalization of all vectors of the set, it is not difficult to show that for III to be satisfied it is sufficient to have $a_1 = Mq_1^2q_2^2 = N^{-2} + o(N^{-5/2})$ as $M \longrightarrow \infty$. For example, for real vectors uniformly distributed on the unit sphere $a_1 = [M(N+1)]^{-1}$. Hence real stochastic vectors uniformly distributed on the unit sphere satisfy III.

b) The set of random vectors having, in some basis, the form $q = N^{-1/2}(\xi_1, \cdots, \xi_N)$, where the ξ_i are identically distributed independent random quantities, with mean value zero, unit dispersion and finite fourth moment μ_4. In this case $a_{ij}(N) = 0$, $\phi_{il}(N) = N^{-1}\delta_{il}$ and among the numbers $b_{ijlm}(N)$ only $b_{iiii}(N) = N^{-2}(\mu_4 - 3)$ differs from zero. Consequently $\epsilon_1(N) = 0$, $\epsilon_2(N) = N^{-1}$, $\epsilon_3(N) = N^{-1/2}|\mu_4 - 3|$, from which it follows that the stochastic vectors of this set also satisfy III.

As we pointed out, the main problem is to prove that the function $\nu(\lambda; c)$ defined by (1.3) exists and to go about finding it. It is more convenient, however, to find instead of $\nu(\lambda; c)$ its Stieltjes transform

$$m(z; c) = \int_{-\infty}^{\infty}\frac{d\nu(\lambda; c)}{\lambda - z}, \quad (1.7)$$

a knowledge of which gives the desired function at all points of continuity by means of the well-known inversion formula

$$\nu(\lambda_2; c) - \nu(\lambda_1; c) = \lim_{\eta\to+0}\frac{1}{\pi}\int_{\lambda_1}^{\lambda_2}\operatorname{Im} m(\xi + i\eta; c)\,d\xi. \quad (1.8)$$

For simplicity we shall denote the function $m(z;\ 0)$, whose existence is ensured by condition II, by $m_0(z)$, so

$$m_0(z) = \int_{-\infty}^{\infty} \frac{d\nu_0(\lambda)}{\lambda - z}, \tag{1.9}$$

where $\nu_0(\lambda)$ is defined by (1.2).

Our basic result is contained in the following theorem.

Theorem 1. *Let I–IV be fulfilled. Then the following assertions are valid.*

1) *The sequence of normalized spectral functions $\nu(\lambda;\ B_N(n))$ of the operators $B_N(n)$ for $N \to \infty$ converges in probability to a nondecreasing function $\nu(\lambda;\ c)$ at all points of continuity. Moreover, $\nu(-\infty;\ c) = \nu_0(-\infty)$, $\nu(+\infty;\ c) = \nu_0(+\infty)$, where the function $\nu_0(\lambda)$ is defined in* (1.2). *Hence at all points of continuity the function $\nu(\lambda;\ c)$ is given in terms of its Stieltjes transform $m(z;\ c)$ by the formula*

$$\nu(\lambda;\ c) = \nu_0(-\infty) + \lim_{\mu\to-\infty}\left\{\lim_{y\to+0}\frac{1}{\pi}\int_{\mu}^{\lambda} \operatorname{Im} m(x+iy;\ c)\,dx\right\}.$$

2) *The Stieltjes transform $m(z;\ c)$ of the function $\nu(\lambda;\ c)$ is equal to the solution at $t = 1$ of the equation*

$$u(z, t) = m_0(z) - c\int_0^t \frac{\tau(\xi)}{1+\tau(\xi)\,u(z,\ \xi)}\,\frac{\partial}{\partial z}u(z,\ \xi)\,d\xi, \tag{1.10}$$

where $m_0(z)$ is defined in (1.9) *and $\tau(\xi)$ (the generalized inverse of the probability distribution function $\sigma(x)$ of the stochastic variable τ) is defined by the formula*[1]

$$\tau(\xi) = \inf_{\tau}\{\tau;\ \sigma(\tau) \geqslant \xi\}. \tag{1.11}$$

3) *The solution to* (1.10) *exists and is unique. Equation* (1.10) *is equivalent to the first order partial differential equation*

$$\frac{\partial u(z,\ t)}{\partial t} + c\frac{\tau(t)}{1+\tau(t)\,u(z,\ t)}\cdot\frac{\partial u(z,\ t)}{\partial z} = 0; \quad u(z, 0) = m_0(z), \tag{1.12}$$

whose solution by the method of characteristics leads to the following implicit expression for $u(z,\ t)$:

$$u(z,\ t) = m_0\left(z - c\int_0^t \frac{\tau(\xi)\,d\xi}{1+\tau(\xi)\,u(z,\ t)}\right). \tag{1.13}$$

To avoid misunderstanding we point out that the solution of equations (1.10) and (1.12) is to be analytic in z and continuous in t in the region $\operatorname{Im} z > 0$ and $t \in [0,\ 1]$.

Note further that from the definition of $\tau(\xi)$ it follows that

1) It is not difficult to verify that $\tau(\xi)$ is a left continuous and nondecreasing function. Where $\tau(\xi)$ and $\sigma(x)$ are strictly increasing, they are inverse to each other. Where $\sigma(x)$ is constant $\tau(\xi)$ has jumps, and where $\sigma(x)$ has jumps $\tau(\xi)$ is piecewise constant.

$$\int_0^1 \frac{\tau(\xi)\, d\xi}{1+\tau(\xi)\, u(z, 1)} = \int_{-\infty}^{\infty} \frac{\tau\, d\sigma(\tau)}{1+\tau u(z, 1)}, \tag{1.14}$$

whence by (1.13), since $m(z; c) = u(z; 1)$, we conclude that the Stieltjes transform $m(z; c)$ of $\nu(\lambda; c)$ satisfies the equation

$$m(z; c) = m_0\left(z - c\int_{-\infty}^{\infty} \frac{\tau\, d\sigma(\tau)}{1+\tau m(z; c)}\right), \tag{1.14}$$

where $\sigma(\tau)$ is the probability distribution for the stochastic variable τ.

We examine three examples.

1) *The sum of random independent and equally probable projections.* Let $B_N(n) = \tau\Sigma_{i=1}^n P_i$, where each P_i is a projection operator on the random vector $q^{(i)}$, independent and uniformly distributed on the unit sphere, and τ is a nonrandom number. It was shown above that III is satisfied in this case. Since $A_N = 0$ and τ is nonrandom, I and IV are also fulfilled and $m_0(z) = -z^{-1}$, $d\sigma(\xi) = \delta(\xi - \tau)d\xi$. Therefore, if $n/N \to c$ as $N \to \infty$, I–IV are also satisfied and $m(z; c)$ satisfies (1.14), which in this case is

$$m(z; c) = -\left(z - \frac{c\tau}{1+\tau m(z; c)}\right)^{-1}.$$

By solving this quadratic equation for $m(z; c)$, we find that

$$m(z; c) = -\frac{(1-c)+|1-c|}{2z} + \frac{-z+|1-c|\tau+\sqrt{(z-c\tau+\tau)^2-4z\tau}}{2z\tau},$$

where for $\operatorname{Im} z > 0$ that branch of the square root must be taken for which $\operatorname{Im} m(z; c) > 0$ (since $m(z; c)$ is the Stieltjes transform of a nondecreasing function).

Hence, using the inversion formula (1.8), we find that $\nu(\lambda; c) = \nu_1(\lambda; c) + \nu_2(\lambda; c)$, where

$$\frac{d\nu_1(\lambda; c)}{d\lambda} = \begin{cases} (1-c)\,\delta(\lambda) & \text{for } 0 \leqslant c \leqslant 1, \\ 0 & \text{for } c > 1, \end{cases}$$

$$\frac{d\nu_2(\lambda; c)}{d\lambda} = \begin{cases} \dfrac{\sqrt{4c\tau^2-(\lambda-c\tau-\tau)^2}}{2\pi\tau\lambda} & \text{for } (\lambda-c\tau-\tau)^2 \leqslant 4c\tau^2, \\ 0 & \text{for } (\lambda-c\tau-\tau)^2 > 4c\tau^2. \end{cases}$$

From these formulas for $c > 1$ it follows, in particular, that the normalized spectral functions of the stochastic operators

$$K_N(n) = -\tau\left(1+\frac{n}{N}\right)I + \tau\sum_{i=1}^{n} P_i \quad \left(\lim_{N\to\infty}\frac{n}{N} = c > 1\right)$$

as $N \to \infty$ converge in probability to the function $\nu(\lambda;\, c)$ whose derivative is

$$\frac{d\nu(\lambda;\, c)}{d\lambda} = \begin{cases} \dfrac{\sqrt{4c\tau^2-\lambda^2}}{2\pi c\tau^2}\left(1+\dfrac{\lambda+\tau}{\tau c}\right)^{-1} & \text{for } \lambda^2 \leqslant 4c\tau^2, \\ 0 & \text{for } \lambda^2 > 4c\tau^2. \end{cases}$$

As $c \to \infty$ the right-hand side of this formula becomes the semicircle law obtained by Wigner for a Gaussian set of random matrices [4]. Of course, this is what is expected since $K_N(n)$ is the sum of independent identically distributed random matrices:

$$K_N(n) = \tau \sum_{i=1}^{n} \left\{-\left(\frac{n+N}{nN}\right) I + P_i\right\},$$

and, by the central limit theorem, the probability distribution for the random matrices $K_N(n)$ should be nearly Gaussian, if $n \gg N$.

This is one of the similarities of our set with Wigner's set explicitly mentioned above.

2) *The sum of random independent and equally probable projections with random bounded coefficients.* Let $B_N(n) = \Sigma_{i=1}^{n} \tau_i P_i$, where the P_i are projections just as before, and the coefficients τ_i are independent identically distributed random quantities with probability density $[\pi(1-\tau^2)^{1/2}]^{-1}$. Conditions I–IV are clearly satisfied if $\lim_{N\to\infty} n/N = c$ exists, if $A_N = 0$, and if

$$m_0(z) = -z^{-1} \quad \text{and} \quad \int_{-\infty}^{\infty} \frac{\tau\, d\sigma(\tau)}{1+\tau u} = \frac{1}{\pi}\int_{-1}^{1} \frac{\tau(1-\tau^2)^{-\frac{1}{2}}}{1+\tau u}\, d\tau = \frac{1}{u}\left[1-(1-u^2)^{-\frac{1}{2}}\right].$$

Therefore in this case (1.14) has the form

$$m(z;\, c) = -\left\{z - \frac{c}{m(z;\, c)}\left[1-(1-m^2(z;\, c))^{-\frac{1}{2}}\right]\right\}^{-1}$$

Hence after some elementary manipulation we obtain

$$z^2m^4 + 2(1-c)\,zm^3 + [(1-c)^2 - z^2]\,m^2 - 2z(1-c)\,m - 1 + 2c = 0,$$

where for simplicity we have set $m(z;\, c) = m$. In particular, for $c = 1$ we obtain a biquadratic equation, which we solve (taking into account that $m(z;\, 1) \to 0$ for $\operatorname{Im} z \to \infty$ and $\operatorname{Im} m(z;\, 1) > 0$ for $\operatorname{Im} z > 0$) by finding $m(z;\, 1)$ and then (by (1.8)) $\nu(\lambda;\, 1)$:

$$\frac{d\nu(\lambda;\, 1)}{d\lambda} = \begin{cases} \dfrac{1}{\pi}\sqrt{\dfrac{2-|\lambda|}{4|\lambda|}} & \text{for } |\lambda| \leqslant 2, \\ 0 & \text{for } |\lambda| > 2. \end{cases}$$

3) *The sum of random independent and equally probable projections with random unbounded coefficients.* Let $B_N(n)$ be the same type of operators as in the preceding example, except that the probability density of the random quantities τ_i we take as $\pi^{-1} a/(a^2+\tau^2)$.

In this case

$$\int_{-\infty}^{\infty} \frac{\tau\, d\sigma(\tau)}{1+\tau u} = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\tau}{1+\tau u}\cdot\frac{a}{a^2+\tau^2}\, d\tau = \frac{-ia}{1-iau},$$

and equation (1.14) has the form

$$m = -\left\{z + \frac{iac}{1 - iam}\right\}^{-1} \quad (m = m(z; c)).$$

By solving this quadratic equation for $m(z; c)$ we obtain, by (1.8), the following expression for $\nu(\lambda; c)$:

$$\nu(\lambda; c) = \nu_1(\lambda; c) + \nu_2(\lambda; c),$$

where

$$\frac{d\nu_1(\lambda; c)}{d\lambda} = \begin{cases} (1-c)\,\delta(\lambda) & \text{for } 0 \leqslant c \leqslant 1, \\ 0 & \text{for } c > 1, \end{cases}$$

$$\frac{d\nu_2(\lambda; c)}{d\lambda} = \frac{\sqrt{\frac{1}{2}\left(\sqrt{(\lambda^2 + \lambda_1^2)(\lambda^2 + \lambda_2^2)} + \lambda^2 - \lambda_1\lambda_2\right) - \lambda}}{2a\lambda}.$$

$$\lambda_{1,2} = a(1 \pm \sqrt{c})^2.$$

From these formulas we see that in this case the spectrum occupies the entire axis, as was to be expected from the unboundedness of r.

We note in conclusion that it is, as a rule, impossible to find $m(z; c)$, and even more so to find $\nu(\lambda; c)$, in explicit form, since (1.14) is, generally speaking, not explicitly solvable for $m(z; c)$. In this sense the above examples are exceptional. Nevertheless, it is frequently possible to obtain a qualitative picture of the spectrum: the number and arrangement of its connected components, and also the behavior of $\nu(\lambda; c)$ near the boundary of the spectrum. We have in mind the following: on the intervals complementary to the spectrum on the real axis the function $m(x + i0; c)$ exists and is continuous, real and montonically increasing. Hence, the inverse function exists on these intervals—also real and monotonically increasing. Furthermore, its range of values is clearly just the complement of the spectrum. By denoting this inverse function by $x(m; c)$ and using (1.14) we obtain

$$x(m; c) = x_0(m; c) + c \int_{-\infty}^{\infty} \frac{\tau\, d\sigma(\tau)}{1 + \tau m}, \tag{1.15}$$

where $x_0(m)$ is the function inverse to $m_0(x)$. Thus we have the following rule for determining the spectrum: it is necessary to locate the intervals where the function on the right-hand side of (1.15) is monotonically increasing and then to determine the set of its values on these intervals. The spectrum is the complement of this set.

Therefore if a is the endpoint of one of the intervals on which the right-hand side of (1.15) is monotonically increasing, then

$$\lambda_a = x_0(a) + c \int_{-\infty}^{\infty} \frac{\tau\, d\sigma(\tau)}{1 + \tau a} \tag{1.16}$$

is the endpoint of one of the connected components of the spectrum. Suppose that in the neighborhood of a the right-hand side of (1.15) is analytic, and consequently has local extremum at this point (a maximum if a is the right endpoint, a minimum if a is the left endpoint of such an interval). Since at a local extremum the first nonvanishing derivative must be of even order, the Taylor expansion of the right-hand side of (1.15) has the form

$$x(m;\, c) = \lambda_a + \frac{d_{2k}}{(2k)!}(m - a)^{2k} + \ldots ,$$

and hence near the point λ_a we have

$$m(z;\, c) - a = \sqrt[2k]{\frac{z - \lambda_a}{d_{2k}}(2k)!}\quad [1 + o(1)],$$

where that branch of the root is taken which has positive imaginary part for $\operatorname{Im} z > 0$ and is real in a neighborhood of λ_a not containing any points of the spectrum. Thus by the inversion formula (1.8) it follows that in a neighborhood of λ_a the function $\nu(\lambda;\, c)$ has a derivative, and as $\lambda \to \lambda_a$

$$\frac{d\nu(\lambda;\, c)}{d\lambda} = \sqrt[2k]{\frac{\lambda_a - \lambda}{d_{2k}}(2k)!}\quad [1 + o(1)]\,\frac{\sin\frac{\pi}{2k}}{\pi}. \tag{1.17}$$

Therefore the rule for finding the singularities of the function $\nu(\lambda;\, c)$ may be formulated as follows.

The boundaries of some of the connected components of the spectrum may be found by formula (1.16), where the local extrema of the right-hand side of (1.15) are taken to be the points a; near these boundary points the function $\nu(\lambda;\, c)$ has algebraic singularities, the principal parts of which are found by using (1.17).

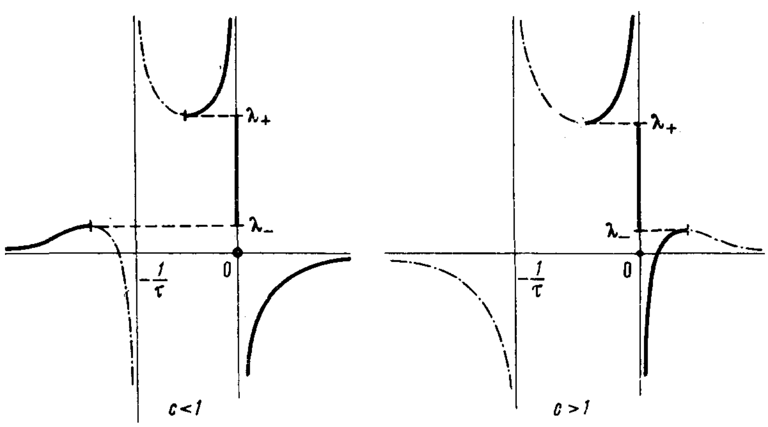

Figure 1

We shall illustrate this rule in the case of example (1) above. Here

$$x(m;\, c) = -\frac{1}{m} + \frac{c\tau}{1 + \tau m}. \tag{1.18}$$

The graph of this function is shown in Figure 1, where the dashes denote where the right-hand side of (1.18) is decreasing (this part must be discarded).

It is clear from the graph that the spectrum consists of the interval $[\lambda_-, \lambda_+]$ and the point 0 for $c < 1$, and the interval $[\lambda_-, \lambda_+]$ alone for $c \geq 1$. The extrema of the right hand side of (1.18) are $m_{\pm} = -1/\tau(1 \pm \sqrt{c})$, from which, according to (1.16), we find the boundaries of the spectrum $\lambda_- = \tau(1-\sqrt{c})^2$ and $\lambda_+ = \tau(1+\sqrt{c})^2$.

The second derivative of the right-hand side of formula (1.18) at the extrema $m_{\pm}$ are

$$d_{\pm}^{(2)} = \pm \frac{2\tau}{\sqrt{c}} \tau^2 (1 \pm \sqrt{c})^2 = \pm \frac{2\tau}{\sqrt{c}} \lambda_{\pm}^2,$$

where by (1.17) it follows that near the points $\lambda_{\pm}$

$$\nu'(\lambda; c) \approx \frac{\sqrt[4]{c}}{\pi\lambda_{\pm}} \sqrt{\frac{|\lambda - \lambda_{\pm}|}{\tau}}$$

in complete agreement with the exact results found above.

§2. Auxiliary considerations

Consider the linear operator A mapping the space H_N into itself. Denote the matrix of A relative to some orthonormal basis in this space by $\|A_{ik}\|$.

Lemma 1. *If the stochastic vector $q = (q_1, q_2, \cdots, q_N)$ obeys condition* III, *then*

$$M|(Aq, q) - N^{-1}\operatorname{Sp} A| \leqslant \|A\| \varepsilon(N),$$

where $\|A\|$ is the norm of A, the quantity $\epsilon(N)$ does not depend on A and tends to zero as $N \to \infty$.

Proof. Letting $\eta = (Aq, q) = \sum_{i,j=1}^N A_{ij}\bar{q}_j\bar{q}_i$ for simplicty, by (1.14) we shall have

$$M\eta = \sum_{i,j} A_{ij} M q_j \bar{q}_i = N^{-1} \sum_{i,j} A_{ij}\delta_{ij} + \sum_{i,j} A_{ij} a_{ji}(N)$$

or

$$|M\eta - N^{-1}\operatorname{Sp} A| = \Big|\sum_{i,j} A_{ij} a_{ji}(N)\Big| \leqslant \Big(\sum_{i,j} |A_{ij}|^2 \cdot \sum_{i,j} |a_{ij}(N)|^2\Big)^{\frac{1}{2}},$$

whence, using the obvious inequality

$$\sum_{i,j} |A_{ij}|^2 \leqslant N \max_i \sum_j |A_{ij}|^2 \leqslant N\|A\|^2, \tag{2.1}$$

we obtain by (1.6) that

$$|M\eta - N^{-1}\operatorname{Sp} A| \leqslant \|A\| \Big(N \sum_{i,j} |a_{ij}(N)|^2\Big)^{\frac{1}{2}} = \|A\| \varepsilon_1(N). \tag{2.2}$$

Similarly, from (1.5) we get

$$M\eta\bar{\eta} = N^{-2}|\operatorname{Sp} A|^2 + N^{-2}\sum A_{ij}\bar{A}_{ij} + \sum A_{ij}\bar{A}_{lm}\varphi_{jl}(N)\overline{\varphi_{im}(N)} + \sum A_{ij}\bar{A}_{lm} b_{jilm}(N).$$

We estimate the second and fourth terms on the right-hand side by means of the inequality (2.1):

$$N^{-2}\sum A_{ij}\bar{A}_{ij}\leqslant N^{-1}\|A\|^2,$$

$$\left|\sum A_{ij}\bar{A}_{lm}b_{jilm}(N)\right|\leqslant\left(\sum|A_{ij}|^2|A_{lm}|^2\right)^{\frac{1}{2}}\left(\sum|b_{ijlm}(N)|^2\right)^{\frac{1}{2}}$$

$$\leqslant\|A\|^2N\left(\sum|b_{ijlm}(N)|^2\right)^{\frac{1}{2}}=\|A\|^2\varepsilon_3(N).$$

We estimate the third term:

$$\left|\sum A_{ij}\bar{A}_{lm}\varphi_{jl}(N)\overline{\varphi_{im}(N)}\right|\leqslant\left(\sum_{i,l}\Big|\sum_j A_{ij}\varphi_{jl}(N)\Big|^2\right)^{\frac{1}{2}}\left(\sum_{i,l}\Big|\sum_m A_{lm}\overline{\varphi_{im}(N)}\Big|^2\right)^{\frac{1}{2}}$$

$$\leqslant\left(\sum_l\|A\|^2\sum_j|\varphi_{jl}(N)|^2\right)^{\frac{1}{2}}\left(\sum_i\|A\|^2\sum_m|\varphi_{im}(N)|^2\right)^{\frac{1}{2}}\leqslant\|A\|^2\sum|\varphi_{jl}(N)|^2$$

$$=\|A\|^2\varepsilon_2(N).$$

Thus,

$$|M\eta\bar{\eta}-N^{-2}|\operatorname{Sp}A|^2|\leqslant\|A\|^2\{N^{-1}+\varepsilon_2(N)+\varepsilon_3(N)\}. \tag{2.3}$$

In addition we have

$$M|\eta-N^{-1}\operatorname{Sp}A|\leqslant\{M|\eta-N^{-1}\operatorname{Sp}A|^2\}^{\frac{1}{2}}$$

$$=\{M\eta\bar{\eta}-N^{-2}|\operatorname{Sp}A|^2-2\operatorname{Re}N^{-1}\overline{\operatorname{Sp}A}(M\eta-N^{-1}\operatorname{Sp}A)\}^{\frac{1}{2}}$$

Hence using inequalities (2.2) and (2.3) and noting that $|N^{-1}\operatorname{Sp}A|\le\|A\|$, we obtain

$$M|\eta-N^{-1}\operatorname{Sp}A|\leqslant\|A\|\{N^{-1}+\varepsilon_2(N)+\varepsilon_3(N)+2\varepsilon_1(N)\}^{\frac{1}{2}}.$$

Therefore, setting $\epsilon(N)=\{N^{-1}+\epsilon_2(N)+\epsilon_3(N)+2\epsilon_1(N)\}^{1/2}$, we have $M|(Aq,q)-N^{-1}\operatorname{Sp}A|\le\|A\|\cdot\epsilon(N)$, where $\epsilon(N)$ does not depend on A and, by condition III, tends to zero as $n\to\infty$. Q.E.D.

Lemma 2. *Let the Hermitian operators $\tilde{A}$ and A, acting in the space H_N, be related by*

$$\tilde{A}-A=\tau(\cdot,q)q,$$

where τ is a real number and q is a stochastic vector which obeys condition III. *Then for the difference of the traces of the resolvents $\tilde{R}_z$ and R_z of these operators we have*

$$\operatorname{Sp}\tilde{R}_z-\operatorname{Sp}R_z=-\frac{\partial}{\partial z}\ln(1+\tau N^{-1}\operatorname{Sp}R_z)+\delta(z,q,N),$$

where $\delta(z,q,N)$ is a random quantity satisfying the inequality

$$M|\delta(z,q,N)|\leqslant2\left|\frac{\tau}{y^2(1+\tau N^{-1}\operatorname{Sp}R_z)}\right|\varepsilon(N),$$

where $\epsilon(N)$ *does not depend on* A, z, τ, *and vanishes as* $N \to \infty$.

Proof. Since the determinant of the matrix of any operator is equal to the product of its eigenvalues, the identity

$$\widetilde{R}_z = (\widetilde{A} - zI)^{-1} = \{(A - zI) + (\widetilde{A} - A)\}^{-1} = \{I + R_z(\widetilde{A} - A)\}^{-1} R_z$$

shows that

$$\prod_1^N (\widetilde{\lambda}_k - z)^{-1} = \prod_1^N (\lambda_k - z)^{-1} \{\det [I + R_z(\widetilde{A} - A)]\}^{-1},$$

where $\widetilde{\lambda}_k$ and λ_k are eigenvalues of $\widetilde{A}$ and A.

Taking the logarithmic derivative with respect to z of both sides of this equation, we find the well-known formula for the difference of the resolvent traces:

$$\operatorname{Sp} \widetilde{R}_z - \operatorname{Sp} R_z = -\frac{\partial}{\partial z} \ln \det [I + R_z(\widetilde{A} - A)].$$

In particular, if $\widetilde{A} - A = \tau(\cdot\,, q)q$ it is easily seen that

$$\det [I + R_z(\widetilde{A} - A)] = 1 + \tau (R_z q, q),$$

and consequently in this case $\operatorname{Sp} \widetilde{R}_z - \operatorname{Sp} R_z = -(\partial/\partial z) \ln [1 + \tau(R_z q, q)]$, or

$$\operatorname{Sp} \widetilde{R}_z - \operatorname{Sp} R_z = -\frac{\tau (R_z^2 q, q)}{1 + \tau (R_z q, q)}. \tag{2.4}$$

We now evaluate the right-hand side of the formula. For this purpose, denote by $E(\lambda)$ the decomposition of the identity for the operator A and introduce the nondecreasing function $\alpha(\lambda) = (E(\lambda)q, q)$. Then

$$(R_z q, q) = \int_{-\infty}^{\infty} \frac{d\alpha(\lambda)}{\lambda - z}, \quad (R_z^2 q, q) = \int_{-\infty}^{\infty} \frac{d\alpha(\lambda)}{(\lambda - z)^2},$$

whence it follows that for $z = x + iy$

$$|1 + \tau (R_z q, q)| \geqslant |\tau \operatorname{Im} (R_z q, q)| = |\tau y| \int_{-\infty}^{\infty} \frac{d\alpha(\lambda)}{(\lambda - x)^2 + y^2},$$

$$|\tau (R_z^2 q, q)| \leqslant |\tau| \int_{-\infty}^{\infty} \frac{d\alpha(\lambda)}{(\lambda - x)^2 + y^2},$$

so that

$$\left| \frac{\tau (R_z^2 q, q)}{1 + \tau (R_z q, q)} \right| \leqslant \frac{1}{|y|}. \tag{2.5}$$

To complete the proof of the lemma we rewrite (2.4) as follows:

$$\operatorname{Sp} \widetilde{R}_z - \operatorname{Sp} R_z = -\frac{\tau N^{-1} \operatorname{Sp} R_z^2}{1 + \tau N^{-1} \operatorname{Sp} R_z} + \delta (z, q, N), \tag{2.6}$$

where

$$\delta(z, q, N) = \frac{\tau N^{-1} \operatorname{Sp} R_z^2}{1 + \tau N^{-1} \operatorname{Sp} R_z} - \frac{\tau (R_z^2 q, q)}{1 + \tau (R_z q, q)},$$

and we evaluate the expectation value $|\delta(z, q, N)|$. We have

$$\delta(z, q, N) = \frac{\tau (R_z^2 q, q)}{1+\tau (R_z q, q)} \cdot \frac{\tau \{(R_z q, q) - N^{-1} \operatorname{Sp} R_z\}}{1 + \tau N^{-1} \operatorname{Sp} R_z} -$$

$$- \frac{\tau}{1 + \tau N^{-1} \operatorname{Sp} R_z} \{(R_z^2 q, q) - N^{-1} \operatorname{Sp} R_z^2\},$$

whence, applying (2.5), we obtain

$$|\delta(z, q, N)| \leqslant \left| \frac{\tau}{y(1 + \tau N^{-1} \operatorname{Sp} R_z)} \right| \{ |(R_z q, q) - N^{-1} \operatorname{Sp} R_z |$$

$$+ |y| \cdot |(R_z^2 q, q) - N^{-1} \operatorname{Sp} R_z| \}.$$

Since the stochastic vector q obeys condition III, from this inequality and Lemma 1 we have

$$M |\delta(z, q, N)| \leqslant \left| \frac{\tau}{y(1 + \tau N^{-1} \operatorname{Sp} R_z)} \right| (\| R_z \| + |y| \cdot \| R_z^2 \|) \varepsilon(N).$$

Since A is Hermitian, it follows that $\|R_z\| \le |y|^{-1}$ and $\|R_z^2\| \le |y|^{-1}$. Therefore

$$M |\delta(z, q, N)| \leqslant 2 \left| \frac{\tau}{y^2 (1 + \tau N^{-1} \operatorname{Sp} R_z)} \right| \varepsilon(N), \tag{2.7}$$

which was to be proved.

Consider now the quantities τ_i. By hypothesis, these are independent quantities with one and the same distribution function $\sigma(x)$. Let T_n be some realization of n of these random quantities. We number the quantities in T_n in order of their size $\tau_1 \le \tau_2 \le \cdots \le \tau_n$ and construct an experimental distribution function $\sigma(x, T_n)$ corresponding to this realization by setting

$$\sigma(x, T_n) = \begin{cases} 0 & \text{for } x \leqslant \tau_1, \\ \dfrac{i}{n} & \text{for } \tau_i \leqslant x < \tau_{i+1}, \\ 0 & \text{for } \tau_n < x. \end{cases}$$

By Glivenko's theorem [5], as $n \to \infty$ the function $\sigma(x, T_n)$ almost certainly converges to $\sigma(x)$ uniformly on the entire axis.

We shall have use for the analogue of this theorem for functions inverse to $\sigma(x, T_n)$ and $\sigma(x)$. Here by the inverse functions we mean functions $\tau(\xi, T_n)$ and $\tau(\xi)$ defined on the interval $(0, 1)$ by

$$\tau(\xi, T_n) = \inf_x \{x: \sigma(x, T_n) \geqslant \xi\}, \tag{2.8}$$

$$\tau(\xi) = \inf_x \{x: \sigma(x) \geqslant \xi\}. \tag{2.8'}$$

Note that as a consequence of the definition of $\sigma(x, T_n)$ we have

$$\tau(\xi, T_n) = \tau_{i+1} \text{ for } \frac{i}{n} \leqslant \xi < \frac{i+1}{n} \quad (i = 0, 1, \ldots, n-1). \tag{2.9}$$

Lemma 3. *If the probability distribution $\sigma(x)$ for the random quantity τ has a first absolute moment, then for $n \to \infty$ the sequence of functions $\tau(\xi, T_n)$ almost certainly converges in the metric of $L^1[0, 1]$ to the function $\tau(\xi)$, i.e. the sequence*

$$\alpha_n = \int_0^1 |\tau(\xi, T_n) - \tau(\xi)|\, d\xi$$

almost certainly converges to zero.

Proof. Note first that by assumption $\int_{-\infty}^{\infty} |x|\, d\sigma(x) < \infty$, and from the definition of $\tau(\xi)$ it follows that $\int_0^1 |\tau(\xi)| d\xi = \int_{-\infty}^{\infty} |x| d\sigma(x) < \infty$. Therefore the function $\tau(\xi)$ is summable on the interval $[0, 1]$.

By Glivenko's theorem the sequence $\beta_n = \sup_{-\infty < x < \infty} |\sigma(x, T_N) - \sigma(x)|$ almost certainly converges to zero. It follows from the definition of β_n that if $\sigma(x, T_n) \geq \xi$ then $\sigma(x) \geq \xi - \beta_n$ and if $\sigma(x) \geq \xi + \beta_n$ then $\sigma(x, T_n) \geq \xi$. Therefore

$$\{x: \sigma(x) \geqslant \xi + \beta_n\} \subset \{x: \sigma(x, T_n) \geqslant \xi\} \subset \{x: \sigma(x) \geqslant \xi - \beta_n\},$$

whence for $\xi \in (\beta_n, 1 - \beta_n)$, by definition of $\tau(\xi, T_n)$ and $\tau(\xi)$, we obtain $\tau(\xi + \beta_n) \geq \tau(\xi, T_n) \geq \tau(\xi - \beta_n)$. Since $\tau(\xi)$ is nondecreasing these inequalities imply that $|\tau(\xi, T_n) - \tau(\xi)| \leq \tau(\xi + \beta_n) - \tau(\xi - \beta_n)$ if $\beta_n \leq \xi \leq 1 - \beta_n$. Hence

$$\int_{\beta_n}^{1-\beta_n} |\tau(\xi, T_n) - \tau(\xi)|\, d\xi \leqslant \int_{\beta_n}^{1-\beta_n} \{\tau(\xi + \beta_n) - \tau(\xi - \beta_n)\}\, d\xi$$
$$= \int_{1-\beta_n}^{1} \tau(\xi)\, d\xi - \int_0^{2\beta_n} \tau(\xi)\, d\xi,$$

and hence $\int_{\beta_n}^{1-\beta_n} |\tau(\xi, T_n) - \tau(\xi)| d\xi$ for $n \to \infty$ almost certainly tends to zero, by Glivenko's theorem and the summability of $\tau(\xi)$. In addition,

$$\int_0^{\beta_n} |\tau(\xi, T_n) - \tau(\xi)|\, d\xi \leqslant \int_0^{\beta_n} |\tau(\xi, T_n)|\, d\xi + \int_0^{\beta_n} |\tau(\xi)|\, d\xi.$$

Let A be any positive number and define the random quantities $\tilde{\tau}_i$ in terms of τ_i as follows:

$$\tilde{\tau}_i = \begin{cases} 0, & \text{if } |\tau_i| < A, \\ |\tau_i|, & \text{if } |\tau_i| \geqslant A. \end{cases} \tag{2.10}$$

Then from (2.9) we obtain

$$\int_0^{\beta_n} |\tau(\xi, T_n)|\, d\xi \leqslant \frac{1}{n} \sum_{i=1}^{n} \tilde{\tau}_i + \int_0^{\beta_n} A\, d\xi$$

and consequently

$$\int_0^{\beta_n} |\tau(\xi) - \tau(\xi, T_n)|\, d\xi \leqslant \frac{1}{n} \sum_{i=1}^{n} \tilde{\tau}_i + \int_0^{\beta_n} \{A + |\tau(\xi)|\}\, d\xi.$$

From the strong law of large numbers we find that as $n \to \infty$ we almost certainly have $n^{-1}\sum_{i=1}^{n}\tilde{\tau}_i \to \int_{|x|\geq A}|x|d\sigma(x)$, whence, by the preceding inequality, Glivenko's theorem and the summability of $\tau(\xi)$, we conclude that, as $n \to \infty$, almost certainly

$$\int_{0}^{\beta_n} |\tau(\xi, T_n) - \tau(\xi)|\, d\xi \leqslant \frac{1}{A} + \int_{|x|\geqslant A} |x|\, d\sigma(x)$$

for any $A > 0$. But this means that $\int_0^{\beta_n}|\tau(\xi, T_n) - \tau(\xi)|d\xi$ almost certainly tends to zero for $n \to \infty$. Similarly, we find that $\int_{1-\beta_n}^{1}|\tau(\xi, T_n) - \tau(\xi)|d\xi$ almost certainly tends to zero as $n \to \infty$. Q.E.D.

Remark. This lemma is required below only for the case where the random quantity τ is bounded.

§ 3. Deduction of the basic equation

We shall prove Theorem 1 by first assuming the random quantities τ_i are bounded, i.e. there is a number $T > 0$ so that any realization of the τ_i satisfies

$$|\tau_i| < T. \tag{3.0}$$

This restriction will be lifted by going to a limit. But in the next two sections (3.0) is assumed to hold.

Suppose that T_n, Q_n is a realization of τ and q entering in (1.1). In this realization we number the pairs τ_i, $q^{(i)}$ in order of increasing τ_i:

$$\tau_1 \leqslant \tau_2 \leqslant \ldots \leqslant \tau_n, \quad q^{(1)} \leqslant q^{(2)} \leqslant \ldots \leqslant q^{(n)}$$

and construct a chain of operators $B_N(i)$ $(i = 1, 2, \cdots, n)$ by setting

$$B_N(i) = A_N + \sum_{\alpha=1}^{i} \tau_\alpha\, q^{(\alpha)}(\cdot, q^{(\alpha)}), \tag{3.1}$$

so that $B_N(0) = A_N$, and the $B_N(n)$ are the operators (1.1) that interest us. Denote the resolvents of the operators $B_N(i)$ by $R_z(i)$. For each realization T_n, Q_n we form the function $u(z, \xi; N, T_n, Q_n)$ defined for all nonreal z and real $\xi \in [0, 1]$ by

$$u(z, \xi; N, T_n, Q_n) = N^{-1}\operatorname{Sp} R_z(i) + nN^{-1}\operatorname{Sp}\{R_z(i+1) - R_z(i)\}\left(\xi - \frac{i}{n}\right)$$
$$\text{for} \quad \xi \in \left[\frac{i}{n}, \frac{i+1}{n}\right] \quad (i = 0, 1, \ldots, n-1). \tag{3.2}$$

Note that $u(z, 0; N, T_n, Q_n)$ and $u(z, 1; N, T_n, Q_n)$ are the Stieltjes transforms of the normalized spectral functions of the operators A_N and $B_N(n)$:

$$u(z, 0; N, T_n, Q_n) = N^{-1}\operatorname{Sp} R_z(0) = \int_{-\infty}^{\infty} \frac{d\nu(\lambda; A_N)}{\lambda - z}, \tag{3.3}$$

$$u(z, 1; N, T_n, Q_n) = N^{-1}\operatorname{Sp} R_z(n) = \int_{-\infty}^{\infty} \frac{d\nu(\lambda; B_N(n))}{\lambda - z}. \tag{3.3'}$$

For the remaining values of $\xi \in [0, 1]$ the function $u(z, \xi; N, T_n, Q_n)$ is the Stieltjes transform of the nondecreasing function

$$\nu(\lambda, \xi; B_N(n)) = [1 - n\xi + i]\,\nu(\lambda; B_N(i)) + [n\xi - i]\,\nu(\lambda; B_N(i+1)),$$
$$\xi \in \left[\frac{i}{n}, \frac{i+1}{n}\right],$$

where $\nu(\lambda, B_N(i))$ denotes the normalized spectral function of the operator $B_N(i)$. By the definition (3.2) of $u(z, \xi, N, T_n, Q_n)$ it is clear that it is continuous in the entire range of definition, holomorphic in z and piecewise linear in ξ.

We shall prove the set of functions $u(z, \xi, N, T_n, Q_n)$ and the set of their derivatives $u_z'(z, \xi, N, T_n, Q_n)$ are compact with respect to uniform convergence in $\xi \in [0, 1]$ and $z \in F$, where F is any bounded set lying a positive distance from the real axis. From the inequalities ($\operatorname{Im} z = y$)

$$|N^{-1}\operatorname{Sp} R_z| \leqslant \frac{1}{|y|}, \quad \left|N^{-1}\frac{d}{dz}\operatorname{Sp} R_z\right| \leqslant \frac{1}{y^2}, \tag{3.4}$$

which hold for the resolvent of any selfadjoint operator, and from (3.2), we have

$$|u(z, \xi; N, T_n, Q_n)| \leqslant \frac{1}{|y|}, \quad |u_z'(z, \xi; N, T_n, Q_n| \leqslant \frac{1}{y^2}. \tag{3.5}$$

Since the operators $B_N(i+1)$ and $B_N(i)$ differ by the one-dimensional operator $\tau_{i+1}(\cdot, q^{(i+1)})q^{(i+1)}$, formula (2.4) can be applied to their resolvents $R_z(i+1)$ and $R_z(i)$. Therefore

$$\operatorname{Sp}\{R_z(i+1) - R_z(i)\} = -\frac{\tau_{i+1}(R_z^2(i)\,q^{(i+1)}, q^{(i+1)})}{1 + \tau_{i+1}(R_z(i)\,q^{(i+1)}, q^{(i+1)})},$$

so by (2.5) we find that

$$|\operatorname{Sp}\{R_z(i+1) - R_z(i)\}| \leqslant \frac{1}{|y|}, \tag{3.6}$$

and consequently

$$\left|\frac{\partial}{\partial\xi}\, u(z, \xi; N, T_n, Q_n)\right| \leqslant \frac{n}{N|y|}. \tag{3.7}$$

Since the function $(\partial/\partial\xi)u(z, \xi, N, T_n, Q_n) = (n/N)\{\operatorname{Sp} R_z(i+1) - \operatorname{Sp} R_z(i)\}$, $i/n \le \xi \le (i+1)/n$, is holomorphic in z, using the Cauchy estimate for the derivative of a holomorphic function at the center of a circle in terms of its maximum modulus on the circumference, we find by (3.6) that

$$\left|\frac{\partial}{\partial\xi}\, u_z'(z, \xi; N, T_n, Q_n)\right| \leqslant \frac{4n}{Ny^2}. \tag{3.8}$$

The inequalities (3.5), (3.7) and (3.8) clearly demonstrate the compactness of the sets $\{u(z, \xi, N, T_n, Q_n)\}$ and $\{u_z'(z, \xi, N, T_n, Q_n)\}$ which we require. Note that so far we have not made use of the boundedness of the $\tilde{\tau}_i$.

We shall now consider the function $u(z, \xi, N, T_n, Q_n)$ for z lying in the halfplane $|\operatorname{Im} z| \ge 3T$, where T is a number bounding the modulus of the τ_i (see (3.0)).

Let $\tau(\xi)$ be the generalized inverse of the probability distribution $\sigma(x)$ for the random quantities τ defined in (2.8), and let G be any bounded set lying in the halfplane $\operatorname{Im} z \geq 3T$.

Lemma 4. *If* I–IV *and* (3.0) *are satisfied, then as* $N \to \infty$ *the expectation value of*

$$\varphi_N = \sup_{\substack{t\in[0,1]\\ z\in G}} \left| u(z;\xi;N,T_n,Q_n) - m_0(z) + c\int_0^t \frac{\tau(\xi)\,u'_z(z,\xi;N,T_n,Q_n)}{1+\tau(\xi)\,u(z,\xi;N,T_n,Q_n)}\,d\xi \right|$$

tends to zero: $\lim_{N\to\infty} M\phi_N = 0$.

Proof. From (3.2) and (3.6) we find that for $\xi \in [i/n,\ (i+1)/n]$

$$|u(z,\xi;N,T_n,Q_n) - N^{-1}\operatorname{Sp} R_z(i)| \leqslant \frac{1}{Ny}. \tag{3.9}$$

Using this inequality and the Cauchy estimate for the derivative of a holomorphic function, we likewise obtain

$$|u'_z(z,\xi;N,T_n,Q_n) - N^{-1}\operatorname{Sp} R_z^2(i)| \leqslant \frac{4}{Ny^2}. \tag{3.10}$$

The operators $B_N(i+1)$ and $B_N(i)$ differ by the one-dimensional operator $\tau_{i+1}(\cdot, q^{(i+1)})q^{(i+1)}$, and consequently Lemma 2 is applicable to their resolvents $R_z(i+1)$ and $R_z(i)$. Using this lemma we may transform (3.2) into the form

$$u(z,\xi;N,T_n,Q_n) = N^{-1}\operatorname{Sp} R_z(i) - \frac{n}{N}\,\frac{\tau_{i+1}N^{-1}\operatorname{Sp} R_z^2(i)}{1+\tau_{i+1}N^{-1}\operatorname{Sp} R_z(i)}\left(\xi - \frac{i}{n}\right)$$

$$+ N^{-1}\theta_i(z)\,\delta_i(z, q^{(i+1)}, N), \tag{3.11}$$

where $0 \leq \theta_i(\xi) = n(\xi - i/n) \leq 1$ and

$$M|\delta_i(z,q^{(i+1)},N)| \leqslant 2\left|\frac{\tau_{i+1}}{y^2[1+\tau_{i+1}N^{-1}\operatorname{Sp} R_z(i)]}\right|\varepsilon(N),$$

and $\epsilon(N) \to 0$ as $N \to \infty$. For $\operatorname{Im} z \geq 3T$ the estimate (3.4) and the inequality $|\tau_{i+1}| \leq T$ imply that

$$|1+\tau_{i+1}\operatorname{Sp} R_z(i)| \geqslant \frac{2}{3}, \tag{3.12}$$

and, in particular,

$$M|\delta_i(z,q^{(i+1)},N)| \leqslant 3T\varepsilon(N). \tag{3.13}$$

By formula (2.9), $\tau_{i+1} = \tau(\xi, T_n)$ for $\xi \in [i/n,\ (i+1)/n]$, where $\tau(\xi, T_n)$ is the function defined in (2.8). Thus for $\xi \in [i/n,\ (i+1)/n]$ we have

$$\frac{n}{N}\frac{\tau_{i+1}N^{-1}\operatorname{Sp}R_z^2(i)}{1+\tau_{i+1}N^{-1}\operatorname{Sp}R_z(i)}\left(t-\frac{i}{n}\right)=\frac{n}{N}\int\limits_{\frac{i}{n}}^{t}\frac{\tau(\xi,T_n)N^{-1}\operatorname{Sp}R_z^2(i)}{1+\tau(\xi,T_n)N^{-1}\operatorname{Sp}R_z(i)}\,d\xi.$$

Replacing $\tau(\xi, T_n)$ on the right side of this equation by $\tau(\xi)$, $N^{-1}\operatorname{Sp}R_z(i)$ by $u(z, \xi; N, T_n, Q_n)$ and $N^{-1}\operatorname{Sp}R_z^2(i)$ by $u_z'(z, \xi; N, T_n, Q_n)$, and estimating the error with the aid of (3.9), (3.10) and (3.12), we find that

$$\left|\frac{n}{N}\frac{\tau_{i+1}N^{-1}\operatorname{Sp}R_z^2(i)}{1+\tau_{i+1}N^{-1}\operatorname{Sp}R_z(i)}\left(t-\frac{i}{n}\right)-\frac{n}{N}\int\limits_{\frac{i}{n}}^{t}\frac{\tau(\xi)u_z'(z,\xi;N,T_n,Q_n)}{1+\tau(\xi)u(z,\xi;N,T_n,Q_n)}\,d\xi\right|$$

$$\leqslant\frac{n}{NT^2}\int\limits_{\frac{i}{n}}^{t}|\tau(\xi,T_n)-\tau(\xi)|\,d\xi+\frac{3}{N^2T},$$

which holds for all $t\in[i/n, (i+1)/n]$ and all z in the halfplane $\operatorname{Im} z\geq 3T$. Since $u(z, i/n; N, T_n, Q_n)=N^{-1}\operatorname{Sp}R_z(i)$, it follows from the last inequality and (3.11) that

$$\left|u(z,t;N,T_n,Q_n)-u\left(z,\frac{i}{n};N,T_n,Q_n\right)+\frac{n}{N}\int\limits_{\frac{i}{n}}^{t}\frac{\tau(\xi)u_z'(z,\xi;N,T_n,Q_n)}{1+\tau(\xi)u(z,\xi;N,T_n,Q_n)}\,d\xi\right|$$

$$\leqslant\frac{n}{NT^2}\int\limits_{\frac{i}{n}}^{t}|\tau(\xi,T_n)-\tau(\xi)|\,d\xi+\frac{3}{N^2T}+\frac{1}{N}\delta_i(z,q^{(i+1)},N)$$

for all $t\in[i/n, (i+1)/n]$ and all z in the halfplane $\operatorname{Im} z\geq 3T$. By combining the inequalities we have obtained, we find for the function

$$\varphi(z,t;N,T_n,Q_n)=u(z,t;N,T_n,Q_n)-m_0(z)+c\int\limits_0^t\frac{\tau(\xi)u_z'(z,\xi;N,T_n,Q_n)}{1+\tau(\xi)u(z,\xi;N,T_n,Q_n)}\,d\xi \tag{3.14}$$

the estimate

$$|\varphi(z,t;N,T_n,Q_n)|\leqslant\frac{n}{NT^2}\int\limits_0^1|\tau(\xi,T_n)-\tau(\xi)|\,d\xi+|m_0(z)-u(z,0;N,T_n,Q_n)|$$

$$+\frac{3}{NT}+\frac{1}{T}\left|c-\frac{n}{N}\right|+\frac{1}{N}\sum_{i=1}^{n-1}|\delta_i(z,q^{(i+1)},N)|.$$

Hence from (3.13) it follows that

$$|\varphi(z,t;N,T_n,Q_n)|\leqslant\frac{n}{NT^2}\int\limits_0^1|\tau(\xi,T_n)-\tau(\xi)|\,d\xi+|m_0(z)-u(z,0;N,T_n,Q_n)|$$

$$+\frac{3}{NT}+\frac{1}{T}\left|c-\frac{n}{N}\right|+\frac{3nT}{N}\varepsilon(N).$$

This inequality, equation (3.3), conditions I, II and Lemma 3 imply that

$$\lim_{N\to\infty} |\varphi(z, t; N, T_n, Q_n)| = 0 \tag{3.15}$$

for each z, t ($\operatorname{Im} z \geq 3T$, $t \in [0, 1]$).

Now note that from (3.5), (3.7) and (3.8) it follows that $u(z, t; N, T_n, Q_n)$, $u_z'(z, t; N, T_n, Q_n)$, and consequently also $\phi(z, t; N, T_n, Q_n)$, are uniformly bounded and equicontinuous. Hence, on the set $\{0 \leq t \leq 1;\ z \in G\}$ for the function $\phi(z, t; N, T_n, Q_n)$ we can find a generalized ϵ-net $t_1, z_1; \cdots; t_{m_\epsilon}, z_{m_\epsilon}$, so that

$$\varphi_N = \sup_{\substack{t\in[0,1]\\ z\in G}} |\varphi(t, z; N, T_n, Q_n)| \leqslant \varepsilon + \max_{1\leqslant i\leqslant m_\varepsilon} |\varphi(t_i, z_i; N, T_n, Q_n)|,$$

and consequently

$$M\varphi_N \leqslant \varepsilon + \sum_{i=1}^{m_\varepsilon} M|\varphi(z_i, t_i; N, T_n, Q_n)|.$$

From this inequality, for any $\epsilon > 0$ we find by (3.15) that $\overline{\lim}_{N\to\infty} M\phi_N \leq \epsilon$, so, since $\epsilon > 0$ is arbitrary, $\lim_{M\to\infty} M\phi_N = 0$, which is what was to be proved.

It clearly follows from the above lemma that there must exist a realization T_n', Q_n' for which

$$\lim_{N\to\infty} \sup_{\substack{t\in[0,1]\\ z\in G}} |\varphi(z, t; N, T_n', Q_n')| = 0. \tag{3.16}$$

We proved above that the sets of functions $u(z, t; N, T_n, Q_n)$ and $u_z'(z, t; N, T_n, Q_n)$ are compact. Therefore from the sequence $u(z, T; N, T_n, Q_n)$ we can select a subsequence which converges to some function $u(z, t)$ uniformly in $t \in [0, 1]$ and $z \in F$, for any closed and bounded set F lying in the upper halfplane. It follows from (3.16) that for $z \in G$ and $t \in [0, 1]$ the function $u(z, t)$ satisfies the equation

$$u(z, t) = m_0(z) - c\int_0^t \frac{\tau(\xi)\, u_z'(z, \xi)}{1 + \tau(\xi)\, u(z, \xi)}\, d\xi. \tag{3.17}$$

Since the function $u(z, \xi)$ is holomorphic in the upper halfplane and has positive imaginary part, both sides of this equation are holomorphic in the upper halfplane and consequently coincide everywhere there. Thus (3.17) has at least one solution continuous in t and z ($t \in [0, 1]$, $\operatorname{Im} z > 0$) and holomorphic in z ($\operatorname{Im} z > 0$) for fixed t.

Let $K(\tau, z_0, R)$ denote the set of functions $f(z, t)$, continuous in z, t lying in some cylinder $0 \leq t \leq 1$, $|z - z_0| \leq R$, holomorphic in z ($|z - z_0| \leq R$) for any $t \in [0, 1]$ and obeying the inequality

$$\sup_{\substack{t\in[0,1]\\ |z-z_0|\leqslant R}} \left| \frac{\tau(t)}{1 + \tau(t) f(z, t)} \right| < \infty. \tag{3.18}$$

We note, for example, that all functions $f(z, t)$ with strictly positive imaginary part belong to the set $K(\tau, z_0, R)$.

By a slight variation of Haar's method [6], we shall show that equation (3.17) can possess only one solution in the set $K(\tau, z_0, R)$. To this end we assume that $\tau(\xi)$ is monotonic, but not necessarily bounded.

Lemma 5. *Equation* (3.17) *cannot have two distinct solutions in the set* $K(\tau, z_0, R)$.

Proof. Let $u_1(z, t)$ and $u_2(z, t)$ be any two solutions to (3.17) which belong to $K(\tau, z_0, R)$. Let ξ_0 be that upper bound of the set of $\xi \in [0, 1]$ which has the property that the difference of the solutions $u_1(z, t) - u_2(z, t) = v(z, t)$ is identically zero for $|z - z_0| \leq R$ and all $t \in [0, \xi]$. We must show that $\xi = 1$. Assume that the opposite is true. Then the function $v(z, t)$ vanishes in the cylinder $0 \leq t \leq \xi_0$, $|z - z_0| \leq R$, but for any $h > 0$ in the cylinder $\xi_0 \leq t \leq \xi_0 + h$, $|z - z_0| \leq R$ there is a point where $v(z, t) \neq 0$. Moreover, from (3.17) we have that

$$v(z, t) = \int_{\xi_0}^{t} [A(z, \xi)\, v(z, \xi) + B(z, \xi)\, v'_z(z, \xi)]\, d\xi \quad \text{for } \xi_0 \leqslant t \leqslant 1,\ |z - z_0| \leqslant R,$$

where the function

$$A(z, \xi) = \frac{c\tau^2(\xi)\, u'_{1z}(z, \xi)}{[1 + \tau(\xi)\, u_1(z, \xi)]\,[1 + \tau(\xi)\, u_2(z, \xi)]}, \quad B(z, \xi) = -\frac{c\tau(\xi)}{1 + \tau(\xi)\, u_1(z, \xi)}$$

holomorphic in z, is uniformly bounded in the cylinder $0 \leq \xi \leq 1$, $|z - z_0| \leq R/2$, as follows from (3.18), which both $u_1(z, \xi)$ and $u_2(z, \xi)$ are assumed to satisfy. Let L denote the upper bound of the modulus of $A(z, \xi)$, $B(z, \xi)$ and consider the function $v(z, t)$ in the cone

$$\leqslant t \leqslant \xi_0 + H,\ |z - z_0| \leqslant (\xi_0 + H - t)\frac{R}{2H}, \tag{3.19}$$

where $H = \min\{\tfrac{1}{2} R/(1 + L),\ 1 - \xi_0\}$. It connot vanish identically in this cone, for then, due to analyticity in z, it would vanish in the cylinder $\xi_0 \leq t \leq \xi_0 + H$, $|z - z_0| \leq R$, which contradicts the definition of ξ_0.

Therefore in this cone the function $e^{-2Lt} v(z, t)$ has a positive maximum value at some point z_1, t_1. For sufficiently small $s > 0$ and any complex α, $|\alpha| \leq L + 1$, the points $z_1 - \alpha s$ and $t_1 - s$ lie in the cone (3.19), and consequently the modulus of the function $e^{-2L(t_1 - s)} v(z_1 - \alpha s, t_1 - s)$ does not exceed the modulus of $e^{-2Lt_1} v(z_1, t_1)$. Since $v(z, t)$ is continuous throughout the initial cylinder, it follows from Cauchy's integral formula for the derivative that $v'_z(z, t)$ is certainly continuous in the cone (3.19). Thus for $s \to 0$

$$v(z_1 - \alpha s, t_1 - s) = v(z_1, t_1 - s) - \alpha s[v'_z(z_1, t_1) + o(1)]. \tag{3.20}$$

In addition, we have

$$v(z_1, t_1) - v(z_1, t_1 - s) = \int_{t_1 - s}^{t_1} [A(z_1, \xi)\, v(z_1, \xi) + B(z_1, \xi)\, v'_z(z_1, \xi)]\, d\xi,$$

whence for $s \to +0$ we get

$$v(z_1, t_1) - v(z_1, t_1 - s) = s\,[Av(z_1, t_1) + Bv_z'(z_1, t_1) + o(1)], \tag{3.21}$$

where $A = \lim_{s\to+0} A(z_1, t_1 - s)$, $B = \lim_{s\to+0} B(z_1, T_1 - s)$. The existence of these limits is guaranteed by the monotonicity of $\tau(\xi)$ and the continuity of $u_i(z, t)$, $u_{iz}'(z, t)$ $(i = 1, 2)$. We have from (3.20) and (3.21) that, for $s \to 0$,

$$\begin{aligned} &e^{-2L(t_1-s)} v(z_1 - \alpha s, t_1 - s) \\ &= e^{-2L(t_1-s)} \{(1 - sA) v(z_1, t_1) + s(\alpha + B) v_z'(z_1, t_1) + s\,o(1)\} \\ &= e^{-2Lt_1} v(z_1, t_1) \left\{1 + 2Ls\left[1 - \frac{A}{2L} - \frac{\alpha + B}{2L} \cdot \frac{v_z'(z_1, t_1)}{v(z_1, t_1)} + o(1)\right]\right\}, \end{aligned}$$

and, if we let $\alpha = -B - e^{i\phi_0}$, where $\phi_0 = \arg(v_z'(z_1, t_1)/v(z_1, t_1))$, then

$$\begin{aligned} &e^{-2L(t_1-s)} v(z_1 - \alpha s, t_1 - s) \\ &= \left\{1 + 2Ls\left[1 - \frac{A}{2L} + \frac{1}{2L}\left|\frac{v_z'(z_1, t_1)}{v(z_1, t_1)}\right| + o(1)\right]\right\} e^{-2Lt_1} v(z_1, t_1). \end{aligned}$$

Since $|A| \leq L$, for small enough $s > 0$ the real part of the expression in curly brackets becomes larger that $1 + 2Lt/3$ and therefore so does its modulus, and hence

$$|e^{-2L(t_1-s)} v(z_1 - \alpha s, t_1 - s)| > |e^{-2Lt_1} v(z_1, t_1)| \tag{3.22}$$

for small enough $s > 0$. On the other hand, since $|B| \leq L$, $|\alpha| \leq L + 1$ and consequently $z_1 - \alpha s$, we see that $t_1 - s$ lies in the cone (3.19) for small $s > 0$, which is inconsistent with (3.22), on whose right-hand side stands the maximum modulus of $e^{-2Lt} v(z, t)$ in this cone. This contradiction shows that the assumption $\xi_0 < 1$ is incorrect. Q.E.D.

§4. Proof of Theorem 1

In the last section we proved the existence of a sequence N' and a realization T_n', Q_n', such that $\lim_{N'\to\infty} u(z, t; N', T_n', Q_n') = u(z, t)$, where $u(z, t)$ is the solution of equation (3.17). Consider the corresponding sequence of normalized spectral functions $\nu(\lambda, t; B_{N'}(n'))$. By Helly's theorems we can pick out a subsequence which converges to a function $\nu'(\lambda)$ at all points of continuity, and by (3.3′) we have

$$\int_{-\infty}^{\infty} \frac{d\nu'(\lambda)}{\lambda - z} = \lim_{N'\to\infty} u(z, 1; N', T_n', Q_n') = u(z, 1).$$

From this formula, for $y \to +\infty$ we have $\nu'(+\infty) - \nu'(-\infty) = -i\lim_{y\to+\infty} yu(iy, 1)$, and we see immediately from (3.17), (3.5) and the definition of $m_0(z)$ in (1.9) that $-i\lim_{y\to\infty} yu(iy, 1) = -i\lim_{y\to+\alpha} ym_0(iy) = \nu_0(+\infty) - \nu_0(-\infty)$. Therefore

$$\nu'(+\infty) - \nu'(-\infty) = \nu_0(+\infty) - \nu_0(-\infty). \tag{4.1}$$

The inversion formula (1.8) allows us to find, corresponding to $u(z, 1)$, the function $\nu'(\lambda)$ exactly to within a constant term which we do not know. In connection with this we introduce the function $\nu(\lambda; c) = \nu'(\lambda) + \nu_0(-\infty)$, which, first of all, has the same Stieltjes transform $u(z, 1)$ as $\nu'(\lambda)$, and, secondly, by (4.1) has the same limits at $\pm\infty$ as $\nu_0(\lambda)$:

$$\int_{-\infty}^{\infty} \frac{d\nu(\lambda; c)}{\lambda - z} = u(z, 1), \tag{4.2}$$

$$\nu(+\infty; c) = \nu_0(+\infty), \quad \nu(-\infty; c) = \nu_0(-\infty). \tag{4.3}$$

Thus by the inversion formula (1.8), at all points of continuity we have

$$\nu(\lambda; c) = \nu_0(-\infty) + \lim_{\mu \to -\infty} \left\{ \lim_{y \to +0} \frac{1}{\pi} \int_{\mu}^{\lambda} \operatorname{Im} u(x + iy, 1)\, dx \right\}. \tag{4.4}$$

To prove the first two assertions in Theorem 1 we must prove that as $N \to \infty$ the sequence $\nu(\lambda; B_N(n))$ converges in probability to $\nu(\lambda; c)$ at all points of continuity. It is not difficult to see that for this it is sufficient to prove that for any $\epsilon > 0$

$$\lim_{N\to\infty} P\{ |\nu(\lambda_1; B_N(n)) - \nu(\lambda_0; B_N(n)) - \nu(\lambda_1; c) + \nu(\lambda_0; c)| < \varepsilon\} = 1, \tag{4.5}$$

$$\lim_{N\to\infty} P\{\nu_0(-\infty) - \varepsilon < \nu(\lambda; B_N(n)) < \nu_0(+\infty) + \varepsilon\} = 1, \tag{4.6}$$

where λ_1 and λ_0 are any two points of continuity of the function $\nu(\lambda; c)$, and λ is any real number. First we prove (4.5) and then (4.6).

Suppose, for simplicity that $\Delta(\lambda; c) = \nu(\lambda; c) - \nu(\lambda_0; c)$, $\Delta(\lambda; B_N(n)) = \nu(\lambda; B_N(n)) - \nu(\lambda_0; B_N(n))$, where λ_0 is some point of continuity of $\nu(\lambda; c)$. Assume that (4.5) is false. Then there is a point λ_1 where $\nu(\lambda; c)$ is continuous, and where for some $\epsilon > 0$

$$\overline{\lim_{N\to\infty}} P\{ |\Delta(\lambda_1; B_N(n)) - \Delta(\lambda; c)| \geqslant \varepsilon\} = \delta > 0$$

and consequently a sequence $N = N_k$ exists for which

$$P\{ |\Delta(\lambda_1; B_N(n)) - \Delta(\lambda; c)| \geqslant \varepsilon\} > \frac{\delta}{2}. \tag{4.7}$$

On the other hand, by Lemma 4, if r is given, a number $N(r)$ can be found such that for $N > N(r)$

$$P\left\{ \sup_{\substack{t \in [0,1] \\ z \in G}} \left| u(z, t; N, T_n, Q_n) - m_0(z) + c\int_0^t \frac{\tau(\xi)\, u_z'(z, \xi; N, T_n, Q_n)}{1 + \tau(\xi)\, u(z, \xi; N, T_n, Q_n)}\, d\xi \right| < \frac{1}{r} \right\} > 1 - \frac{\delta}{4}. \tag{4.8}$$

Hence a subsequence N_r' can be selected from the sequence N_k such that (4.7) and (4.8) are both satisfied for $N = N_r'$. Since one always has $P\{\mathfrak{A} \cap \mathfrak{B}\} \geq P\{\mathfrak{A}\} + P\{\mathfrak{B}\} - 1$, for $N = N_r'$ the probability of fulfilling the inequalities

$$|\Delta(\lambda_1; B_N(n)) - \Delta(\lambda_1; c)| \geqslant \varepsilon, \tag{4.9}$$

$$\sup_{\substack{t\in[0,1]\\ z\in G}} \left| u(z, t; N, T_n, Q_n) - m_0(z) + c\int_0^t \frac{\tau(\xi)\, u_z'(z, \xi; N, T_n, Q_n)}{1+\tau(\xi)\, u(z, \xi; N, T_n, Q_n)}\, d\xi \right| < \frac{1}{r} \tag{4.10}$$

at the same time is not less than $\delta/2 + 1 - \delta/4 - 1 = \delta/4 > 0$. Hence for all $N = N_r'$ realizations T_n'', Q_n'' must exist for which (4.9) and (4.10) are both satisfied.

From the compactness of the sets $\{u(z, \xi, N, T_n, Q_n)\}$ and $\{u_z'(z, \xi, N, T_n, Q_n)\}$ and from Helly's theorems it follows that a subsequence N_r'' can be selected from N_r' such that $\lim_{N_r''\to\infty} u(z, t; N_r'', T_n'', Q_n'') = u_1(z, t)$ uniformly for $t \in [0, 1]$, $z \in G$ and at all points of continuity $\lim_{N_r''\to\infty} \nu(\lambda; B_{N_r''}(n'')) = \widetilde{\nu}(\lambda; c)$. Moreover, by (4.9)

$$|\widetilde{\nu}(\lambda_1; c) - \widetilde{\nu}(\lambda_0; c) - \nu(\lambda_1; c) + \nu(\lambda_0; c)| \geqslant \varepsilon, \tag{4.11}$$

by (4.10) the function $u_1(z, t)$ satisfies (3.17), and by (3.3′) $\int_{-\infty}^{\infty} (\lambda - z)^{-1} d\widetilde{\nu}(\lambda; z) = u_1(z, 1)$.

Since (3.17) cannot have two different solutions (Lemma 5), $u_1(z, t) = u(z, t)$, so for $t = 1$ we have

$$\int_{-\infty}^{\infty} \frac{d\widetilde{\nu}(\lambda; c)}{\lambda - z} \equiv u(z, 1) \equiv \int_{-\infty}^{\infty} \frac{d\nu(\lambda; c)}{\lambda - z},$$

which contradicts (4.11). Consequently the assumption we made is incorrect and (4.5) is valid.

We turn to the proof of equation (4.6). The random parts of the operators $B_N(n)$ clearly obey the inequalities[1)]

$$-\sum_{i=1}^{n} |\tau_i|(\cdot, q^{(i)})q^{(i)} \leqslant \sum_{i=1}^{n} \tau_i(\cdot, q^{(i)})q^{(i)} \leqslant \sum_{i=1}^{n} |\tau_i|(\cdot, q^{(i)})q^{(i)}. \tag{4.12}$$

Let $E(\lambda)$ be the decomposition of unity corresponding to the nonnegative operator $D = \sum_{i=1}^{n} |\tau_i|(\cdot, q^{(i)})q^{(i)}$, let l be an arbitrary positive number and let $D_1 = \int_0^l \lambda dE(\lambda)$, $D_2 = \int_l^\infty \lambda dE(\lambda)$ so that $D = D_1 + D_2$. It is clear that $\|D_1\| \leq l$ and the number of nonzero eigenvalues (i.e. the dimension of the range) of the operator D_2 is $N - \mathrm{Sp}\, E(l)$. Taking this into account in (4.12), we may write for $B_N(n)$ the inequality

$$A_N - lI - D_2 \leqslant B_N(n) \leqslant A_N + lI + D_2. \tag{4.13}$$

The normalized spectral function of the operators $A_N \pm lI$ is $\nu(\lambda \pm l; A_N)$, where $\nu(\lambda; A_N)$ is the normalized spectral function for the operator A_N. Since the addition of D_2 may change the number of eigenvalues lying in some interval by not more than the dimension of the range of D_2, the normalized

1) We write $A \leq B$ for Hermitian matrices if all the eigenvalues of $B - A$ are nonnegative.

spectral function of the operator on the left (right) side of (4.13) is no larger than $\nu(\lambda + l;\ A_N) + (N - \operatorname{Sp} E(l))/N$ (no less than $\nu(\lambda - l;\ A_N) - (N - \operatorname{Sp} E(l))/N$). Thus for the normalized spectral function $\nu(\lambda;\ B_N(n))$ of the operator $B_N(n)$ we have the estimate

$$\nu(\lambda - l;\ A_N) - [1 - N^{-1}\operatorname{Sp} E(l)] \leqslant \nu(\lambda;\ B_N(n)) \leqslant \nu(\lambda + l;\ A_N) + [1 - N^{-1}\operatorname{Sp} E(l)]. \tag{4.14}$$

Furthermore,

$$\operatorname{Sp} D = \sum_{i=1}^{n} |\tau_i| (q^{(i)}, q^{(i)}) = \int_0^\infty \lambda d \operatorname{Sp} E(\lambda) \geqslant l \int_l^\infty d \operatorname{Sp} E(\lambda) = l[N - \operatorname{Sp} E(l)],$$

so that

$$1 - N^{-1}\operatorname{Sp} E(l) \leqslant \frac{1}{l}\cdot\frac{1}{N}\sum_{i=1}^{n} |\tau_i| (q^{(i)}, q^{(i)}),$$

whence, since τ is bounded (condition (3.0)), we find that

$$1 - N^{-1}\operatorname{Sp} E(l) \leqslant \frac{cT}{2l}\cdot\frac{1}{n}\sum_{i=1}^{n} (q^{(i)}, q^{(i)}).$$

It follows from III that the random quantity $(q^{(i)}, q^{(i)})$ has expectation value no larger than $[1 + \epsilon_1(N)]^{-1}$ and dispersion no greater than $[2 + \epsilon_2(N) + \epsilon_3(N)]^{-1}$. Since the random quantities $(q^{(i)}, q^{(i)})$ are independent, we find from the Čebyšev inequality that $P\{n^{-1}\sum_{i=1}^{n}(q^{(i)}, q^{(i)}) > 2\} \le 3/n$ and consequently $P\{[1 - N^{-1}\operatorname{Sp} E(l)] \le cT/l\} \ge 1 - 3/n$. Hence we conclude from (4.14) that

$$\lim_{N\to\infty} P\left\{\nu(\lambda - l;\ A_N) - \frac{cT}{l} \leqslant \nu(\lambda;\ B_N(n)) \leqslant \nu(\lambda + l;\ A_N) + \frac{cT}{l}\right\} = 1,$$

and since by assumption $\nu(\lambda;\ A_N) \to \nu_0(\lambda)$ for $N \to \infty$ and $\nu_0(-\infty) \le \nu_0(\lambda) \le \nu_0(+\infty)$, we get

$$\lim_{N\to\infty} P\left\{\nu_0(-\infty) - \frac{cT}{l} \leqslant \nu(\lambda;\ B_N(n)) \leqslant \nu_0(+\infty) + \frac{cT}{l}\right\} = 1,$$

whence, since $l > 0$ is arbitrary, we get the necessary formula (4.6).

Thus the first two assertions of Theorem 1 are proved. The existence and uniqueness of the solution of (3.17) was established earlier. The equivalence of this equation to the partial differential equation (1.12) is evident. As for (1.13), it is simplest to verify it directly.

Thus Theorem 1 is proved completely for the case where the τ_i are bounded.

The extension of this theorem to the case of arbitrary random quantities will be given in the following paragraph.

In connection with this theorem we note that

1) The numbers $\nu(-\infty;\ c)$ and $\nu(+\infty;\ c)$ are equal to the relative number of eigenvalues of the operators $B_N(n)$ going off to $-\infty$ and $+\infty$, respectively. As was shown earlier, they are also equal

to the relative number of eigenvalues of the operators A_N going off to $-\infty$ and $+\infty$.

2) Instead of I, it is enough to require the convergence of $\nu(\lambda;\ A_N)$ to $\nu_0(\lambda)$ in probability, assuming that A_N is a random operator not depending on τ_i, $q^{(i)}$.

3) It can be shown that the solution of (1.13) can also be found by the method of successive approximations wherein each approximation has positive imaginary part for $\operatorname{Im} z > 0$.

§5. Generalization

Now let the τ be arbitrary independent stochastic variables obeying the same probability distribution $\sigma(x)$. At the same time we consider stochastic variables τ^T, defined as follows:

$$\tau^T = \begin{cases} -T, & \text{if} \quad \tau < -T, \\ \tau, & \text{if} \quad -T \leqslant \tau \leqslant T, \\ T, & \text{if} \quad T < \tau, \end{cases}$$

where T is an arbitrary positive number. The probability distribution $\sigma^T(x)$ for the τ_i^T and its inverse $\tau^T(\xi) = \inf_x \{x\colon \sigma^T(x) \geq \xi\}$ are expressed in terms of $\sigma(x)$ and $\tau(\xi) = \inf_x\{x\colon \sigma(x) \geq \xi\}$ as follows:

$$\sigma^T(x) = \begin{cases} 0 & \text{for } x \leqslant -T, \\ \sigma(x) & \text{for } -T < x \leqslant T, \\ 1 & \text{for } T < x, \end{cases} \qquad \tau^T(\xi) = \begin{cases} -T & \text{for } 0 < \xi \leqslant \sigma(-T), \\ \tau(\xi) & \text{for } \sigma(-T) < \xi \leqslant \sigma(T), \\ T & \text{for } \sigma(T) < \xi \leqslant 1. \end{cases} \tag{5.1}$$

Let T_n, Q_n be a realization of the random quantities τ_j (and consequently also of τ_j^T) and vectors $q^{(j)}$. Just as in §3, we form chains of operators $B_N(i)$ and $B_N^T(i)$. It is clear that the $B_N^T(i)$ are obtained from $B_N(i)$ by replacing τ_j by τ_j^T in formula (3.1). Therefore $B_N(i) - B_N^T(i) = \Sigma_{\alpha=1}^n (\tau_\alpha - \tau_\alpha^T) q^{(\alpha)}(\cdot, q^{(\alpha)})$, whence it is clear that the dimension of the range of $B_N(i) - B_N^T(i)$ does not exceed the number of those τ_j whose absolute magnitude is larger than T, i.e. the numbers $n\{\sigma(-T, T_n) + 1 - \sigma(T, T_n)\}$, where $\sigma(x, T_n)$ is the experimental distribution function constructed with reference to the realization T_n of the τ_j. Therefore the normalized spectral functions $\nu(\lambda;\ B_N(i))$ and $\nu(\lambda;\ B_N^T(i))$ of $B_N(i)$ and $B_N^T(i)$ satisfy the inequality

$$|\,\nu(\lambda; B_N(i)) - \nu(\lambda; B_N^T(i))\,| \leqslant \frac{n}{N}\{\sigma(-T, T_n) + 1 - \sigma(T, T_n)\}. \tag{5.2}$$

In addition, let $u(z,\ \xi;\ N,\ T_n,\ Q_n)$ and $u^T(z,\ \xi;\ N,\ T_n,\ Q_n)$ be the functions constructed according to formula (3.2) for the chains of operator $B_N(i)$ and $B_N^T(i)$ respectively. It follows by definition, and from (5.2), that, uniformly in z and ξ,

$$|\,u(z, \xi; N, T_n, Q_n) - u^T(z, \xi; N, T_n, Q_n)\,| \leqslant \frac{1}{\pi y} \cdot \frac{n}{N}\{\sigma(-T, T_n) + 1 - \sigma(T, T_n)\}. \tag{5.3}$$

Since the random quantities τ_j^T are bounded the results of the preceding section are all applicable to the function $u^T(z,\ \xi;\ N,\ T_n,\ Q_n)$, so that, in particular, for $N \to \infty$ it converges in probability to a function $u^T(z,\ t)$ obeying the equation

$$u^T(z,t)=m_0(z)-c\int_0^t \frac{\tau^T(\xi)\,u^{T'}_z(z,\xi)}{1+\tau^T(\xi)\,u^T(z,\xi)}\,d\xi. \tag{5.4}$$

By Glivenko's theorem, as $N\to\infty$ the right-hand side of (5.3) almost certainly tends to $(c/\pi y)\{\sigma(-T)+1-\sigma(T)\}$. Therefore if $T_1>T$ we get

$$|u^{T_1}(z,t)-u^T(z,t)|\leqslant\frac{c}{\pi y}\{\sigma(-T)-\sigma(-T_1)+\sigma(T_1)-\sigma(T)\},$$

whence it is clear that as $T\to\infty$ the function $u^T(z,t)$ tends to some function $u(z,t)$ uniformly for $t\in[0,1]$ and z lying in the region $\operatorname{Im} z>\delta$, where δ is any positive number.

As in the proof of (2.5) it is shown that

$$\left|\frac{\tau^T(\xi)\,u_z^{T'}(z,\xi)}{1+\tau^T(\xi)\,u^T(z,\xi)}\right|\leqslant\frac{1}{|y|}\quad(y=\operatorname{Im} z).$$

This inequality allows us to go to the limit under the integral sign in (5.4) as $T\to\infty$. Since $u^T(z,t)\to u(z,t)$ and $u_z^{T'}(z,t)\to u_z'(z,t)$ as $T\to\infty$, by (5.1), taking the limit in (5.4), we find that the function $u(z,t)$ satisfies the equation

$$u(z,t)=m_0(z)-c\int_0^t\frac{\tau(\xi)\,u_z'(z,\xi)}{1+\tau(\xi)\,u(z,\xi)}\,d\xi. \tag{5.5}$$

Furthermore, we have

$$|u(z,t;N,T_n,Q_n)-u(z,t)|\leqslant|u(z,t;N,T_n,Q_n)-u^T(z,t;N,T_n,Q_n)|$$
$$+|u^T(z,t;N,T_n,Q_n)-u^T(z,t)|+|u^T(z,t)-u(z,t)|,$$

whence in view of (5.3) we conclude that as $N\to\infty$ the function $u(z,t;N,T_n,Q_n)$ converges in probability to $u(z,t)$. It follows immediately from (5.4) that $-\lim_{y\to\infty}y\operatorname{Im}u^T(z,t)=-\lim_{y\to\infty}y\operatorname{Im}m_0(z,t)=\nu_0(+\infty)-\nu_0(-\infty)$, so that, using (5.2), we conclude that $-\lim_{y\to\infty}y\operatorname{Im}u(z,t)=\nu_0(\infty)-\nu_0(-\infty)>0$, and consequently $\operatorname{Im}u(z,t)\neq 0$ for all $t\in[0,1]$ and z lying strictly within the upper halfplane. This implies that $u(z,t)$ satisfies (3.18).

We have therefore shown that as $N\to\infty$ the sequence of functions $u(z,t;N,T_n,Q_n)$ converges in probability to $u(z,t)$, which is the unique solution to equation (5.5) which satisfies (3.18). Thus all the results of §3 are generalized to the case where τ is unbounded. Hence, just as in §4, we conclude that the increment $\nu(\lambda_1;B_N(n))-\nu(\lambda_0;B_N(n))$ as $N\to\infty$ tends in probability to

$$\lim_{\eta\to+0}\frac{1}{\pi}\int_{\lambda_0}^{\lambda_1}\operatorname{Im}u(\xi+i\eta;1)\,d\xi,$$

and from (5.2) and the results of §4 it follows that for any $\epsilon>0$

$$\lim_{N\to\infty}P\{\nu_0(-\infty)-\varepsilon\leqslant\nu(\lambda;B_N(n))\leqslant\nu_0(+\infty)+\varepsilon\}=1.$$

Therefore Theorem 1 is also proved for unbounded τ.

In conclusion we formulate some results, similar to Theorem 1, for two additional sets of random matrices.

Let $q^{(i)}$ $(i = 1, 2, \cdots, n)$ be independent, identically distributed random unit vectors and let $\tau(t)$ $(0 \le t \le 1)$ be continuous real functions for which $\sup_{t\in[0,1]}\tau(t) > -1$. Consider Hermitian matrices $B_N(n)$ of the form

$$B_N(n) = (I + \tau(1)P_n)\left(I + \tau\left(\frac{n-1}{n}\right)P_{n-1}\right)$$
$$\cdots\left(I + \tau\left(\frac{1}{n}\right)P_1\right)A_N\left(I + \tau\left(\frac{1}{n}\right)P_1\right)\cdots(I + \tau(1)P_n),$$

where $P_k = (\cdot, q^{(k)})q^{(k)}$ is the projection onto $q^{(k)}$ and A_N is some random Hermitian operator.

Theorem 2. *If conditions* I–III *from* §1 *are satisfied then the normalized spectral functions for the operators* $B_N(n)$ *as* $N \to \infty$ *tend in probability to some limit function* $\nu(\lambda; c)$ *at all of its points of continuity. The Stieltjes transformation of the function* $\nu(\lambda; c)$ *is equal to the solution of the equation*

$$\frac{\partial u(z,t)}{\partial t} + c\frac{\partial}{\partial z}\ln[1 - \alpha(t)zu(z,t)] = 0,$$

$$u(z,0) = m_0(z) = \int_{-\infty}^{\infty}\frac{d\nu_0(\lambda)}{\lambda - z}, \qquad \alpha(t) = \frac{\tau(t)[2 + \tau(t)]}{[1 + \tau(t)]^2},$$

evaluated at $t = 1$, *where* $\nu_0(\lambda)$ *is the limit of the normalized spectral function of* A_N, *whose existence is ensured by condition* II. *The solution to this equation exists, is unique (in the class of functions having positive imaginary part for* $\operatorname{Im} z > 0$) *and is implicitly given by the formula*

$$u(z,t) = z^{-1}wm_0(w), \quad \text{где } w = z\exp\left\{c\int_0^t\frac{\alpha(\xi)\,d\xi}{1 - \alpha(\xi)zu(z,t)}\right\}.$$

The values of $\nu(\lambda; c)$ *at* $\pm\infty$ *equal those of* $\nu_0(\lambda)$: $\nu(\pm\infty; c) = \nu_0(\pm\infty)$.

Consider next the set of unitary matrices

$$U_N(n) = V_N\prod_{k=1}^{n}\left[(I - P_k) + P_k\exp\left\{i\tau\left(\frac{k}{n}\right)\right\}\right],$$

where P_k is the same projection as above, $\tau(t)$ is a function continuous in the segment $[0, 2\pi]$, V_N is a nonrandom unitary operator and the factors in the product are arranged in order of increasing k.

The normalized spectral function of a unitary operator is the function $\nu(\lambda)$ $(0 < \lambda \le 2\pi)$ equal to the number of eigenvalues of this operator lying on the arc $0 \le \phi < \lambda$ of the unit circle divided by the dimension of the space. Rather than the Stieltjes transform it is more natural here to consider the function

$$n(z) = \int_0^{2\pi}\frac{e^{i\lambda} + z}{e^{i\lambda} - z}\,d\nu(\lambda), \tag{5.6}$$

relative to which $\nu(\lambda)$ is found by the inversion formula

$$\nu(\lambda_2) - \nu(\lambda_1) = \lim_{r \to 1-0} \frac{1}{2\pi} \int_{\lambda_1}^{\lambda_2} \operatorname{Re} n(re^{-i\theta})\, d\theta.$$

Theorem 3. *If as $N \to \infty$ the normalized spectral function of V_N tends to a function $\nu_0(\lambda)$ at all of its points of continuity and conditions* I *and* III *of* §1 *are satisfied, then the sequence of normalized spectral functions of the operators $U_N(n)$ converges in probability as $N \to \infty$ to a function $\nu(\lambda; c)$ at all of its points of continuity. The function $n(z; c)$ corresponding to $\nu(\lambda; c)$ by* (5.6) *is the solution to the equation*

$$\frac{\partial u(z,t)}{\partial t} + 2z \frac{\partial}{\partial z} \ln \left[1 + iu(z,t) \operatorname{tg} \frac{\tau(t)}{2} \right], \quad u(z,0) = n_0(z) = \int_0^{2\pi} \frac{e^{i\lambda} + z}{e^{i\lambda} - z}\, d\nu_0(\lambda).$$

evaluated at $t = 1$. This equation has a unique solution (in the class of functions with positive real part for $|z| < 1$), given implicitly by the formula

$$u(z,t) = n_0 \left(z \exp \left\{ -2c \int_0^t \frac{i \operatorname{tg} \frac{\tau(\xi)}{2}}{1 + iu(z,t) \operatorname{tg} \frac{\tau(\xi)}{2}}\, d\xi \right\} \right).$$

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Translated by:
M. L. Glasser